

# Comment on “Long Time Evolution of Phase Oscillator Systems” [Chaos 19, 023117 (2009)]

Edward Ott, Brian R. Hunt and Thomas M. Antonsen  
University of Maryland, College Park, MD 20742

Reference [1] (henceforth referred to as *I*) considered a general class of problems involving the evolution of large systems of globally coupled phase oscillators. It was shown there that, in an appropriate sense, the solutions to these problems are time asymptotically attracted toward a reduced manifold of system states (denoted  $M$ ). This result has considerable utility in the analysis of these systems, as has been amply demonstrated in recent papers [2]-[13]. In this note, we show that the analysis of *I* can be modified in a simple way that establishes significant extensions of the range of validity of our previous result. In particular, we generalize *I* in the following ways: (1) attraction to  $M$  is now shown for a very general class of oscillator frequency distribution functions  $g(\omega)$ , and (2) a previous restriction on the allowed class of initial conditions is now substantially relaxed. In particular, with respect to point (1), we now show that the result of *I* (derived there for the special case of Lorentzian  $g(\omega)$ ) actually applies for a very general distributions  $g(\omega)$  described in the third paragraph below.

To proceed, we recall the general class of problems described by Eqs.(1) to (4) of *I*, where the time evolution of the quantity  $H(t)$  [appearing in Eq.(4) of *I*] is determined from the time evolution of the order parameter  $r(t)$  [given by Eq.(2) of *I*], and, as discussed in *I*, this determination of  $H$  from  $r$  can arise in various ways depending on the particular problem under consideration. The oscillator distribution function  $F(\theta, \omega, t)$  is then expressed [Eqs.(5) and (6) of *I*] in terms of a decomposition involving functions  $F_+(\theta, \omega, t)$  and  $F_-(\theta, \omega, t)$ , where the analytic continuation of  $F_+$  ( $F_-$ ) into  $Im(\theta) > 0$  ( $Im(\theta) < 0$ ) has no singularities and decays to zero as  $Im(\theta) \rightarrow +\infty$  ( $Im(\theta) \rightarrow -\infty$ ).  $F_+$  is further decomposed [Eq.(8) of *I*] into two parts,  $F_+ = \hat{F}_+ + \hat{F}'_+$ , where  $\hat{F}'_+$  lies on the reduced manifold  $M$  and has dynamics described by Eq.(9) of *I*, while  $\hat{F}_+$  satisfies

$$\frac{\partial \hat{F}_+}{\partial t} + \frac{\partial}{\partial \theta} \left\{ \left[ \omega + \frac{1}{2i} (He^{-i\theta} - H^* e^{i\theta}) \right] \hat{F}_+ \right\} = 0. \quad (1)$$

As argued in *I*, the time asymptotic evolution of the order parameter will be completely described by dynamics on  $M$  (i.e., by  $\hat{F}'_+$ ) provided that

$$\lim_{t \rightarrow +\infty} \int_{-\infty}^{+\infty} \hat{F}_+(\theta, \omega, t) g(\omega) d\omega = 0, \quad (2)$$

where  $g(\omega)$  is the oscillator frequency distribution function. In *I* it is shown that Eq.(2) is satisfied for the case where  $g(\omega)$  is Lorentzian and the analytic continuation of the initial condition,  $\hat{F}_+(\theta, \omega, 0)$ , into  $Im(\omega) < 0$  has no singularities and approaches zero as  $|\omega| \rightarrow \infty$ ,  $Im(\omega) < 0$ . Here we will show that (2) is satisfied much more generally.

We consider a general class of oscillator frequency distributions  $g(\omega)$  that are analytic for real  $\omega$  and can be analytically continued into a strip  $S$  defined by  $0 \geq Im(\omega) > -\sigma$ ,  $\sigma > 0$ , in which  $g(\omega)$  has no singularities and decays to zero as  $Re(\omega) \rightarrow \pm\infty$  fast enough so that  $\int_{-\infty}^{+\infty} g(\omega_r + i\omega_i) d\omega_r$  is well-defined. Some examples of  $g(\omega)$  satisfying these conditions are the Maxwellian,  $g(\omega) \sim \exp\{-(\omega - \bar{\omega})^2/[2(\Delta\omega)^2]\}$ , the Lorentzian,  $g(\omega) \sim [(\omega - \bar{\omega})^2 + (\Delta\omega)^2]^{-1}$ , and the sech distribution,  $g(\omega) \sim \text{sech}[(\omega - \bar{\omega})/\Delta\omega]$ . In contrast, paper *I* restricted consideration to Lorentzian  $g(\omega)$ . We further assume that the initial condition  $\hat{F}_+(\theta, \omega, 0)$  satisfies the same condition in  $S$  as  $g(\omega)$ . Note that this condition on  $\hat{F}_+(\theta, \omega, 0)$  is considerably weaker than the condition used in *I* where it was required that  $\hat{F}_+(\theta, \omega, 0)$  be analytic everywhere in  $Im(\omega) < 0$  and  $\hat{F}_+(\theta, \omega, 0) \rightarrow 0$  for  $|\omega| \rightarrow \infty$ ,  $Im(\omega) < 0$ .

In order to show (2) for the above conditions on  $g$  and  $\hat{F}_+$ , we consider the solution of Eq.(1) for  $\omega$  in the strip  $S$ . Noting that  $\omega$  only appears as a parameter in (1) we can, for the moment, regard  $\omega = \omega_r + i\omega_i$  ( $\omega_i < 0$ ) as fixed. Furthermore, by means of the replacement,  $\hat{F}_+ \rightarrow \hat{F}_+ e^{-i\omega_r t}$ , we can, without loss of generality, take  $\omega_r = 0$ . Thus Eq.(1) takes the same form as Eqs.(17) and (18) of *I*. Furthermore, it was shown [see Eqs.(19) to (31) of *I* and the accompanying discussion] that the solution of (18) and (19) of *I* approaches zero at  $t \rightarrow +\infty$ . Thus, for any  $\omega \in S$  and  $Im(\omega) < 0$ , we have that  $|\hat{F}_+| \rightarrow 0$  as  $t \rightarrow +\infty$ . To complete our argument, we now note that our conditions allow us to shift the integration path for the integral in (2) from the real  $\omega$ -axis to the line  $\omega_r + i\omega_i$ ,  $0 > \omega_i > -\sigma$ ,  $-\infty \leq \omega_r \leq +\infty$ . Since at every point  $\omega_r$  on this line  $|g(\omega)|$  is bounded (because  $g(\omega)$  has no singularities in  $S$ ) and  $\hat{F}_+ \rightarrow 0$  as  $t \rightarrow \infty$ , we conclude that Eq.(2) is satisfied.

The above establishes a greatly expanded range of situation in which Eq.(9) of  $I$  can be used for discovering the time asymptotic dynamics of phase oscillator systems. We note, however, that, while (9) of  $I$  is substantially simpler than the original problem, it is still nontrivial. Thus for the solution of (9) for the order parameter dynamics it may still be useful to employ special choices for  $g(\omega)$  such as the Lorentzian (see Ref.[2]).

Finally, we note that our requirement that  $g(\omega)$  be analytic means that our proof of attraction to  $M$  does not apply if  $g(\omega)$  is a delta function. Thus attraction to  $M$  necessitates that  $g(\omega)$  have a finite width. This is in line with the intuition that relaxation to  $M$  results from phase mixing of oscillators in a population with heterogeneous frequencies. The inapplicability of our result in the case where oscillators all have the same frequency is also consistent with past results [9, 14, 15] which imply that, when there is no frequency spread, the dynamics is not attracted to  $M$ .

This work was supported by ONR through grant N00014-07-1-0734.

- 
- [1] E. Ott and T.M. Antonsen, Chaos **19**, 023117 (2009).
  - [2] E. Ott and T.M. Antonsen, Chaos **18**, 037113 (2008).
  - [3] C.R. Laing, Physica D **238**,1569 (2009).
  - [4] W.S. Lee, E. Ott and T.M. Antonsen, Phys.Rev.Lett. **103** 044101 (2009).
  - [5] C.R. Laing, Chaos **19**, 013113 (2009).
  - [6] M.M. Abdulrehem and E. Ott, Chaos **19**, 013129 (2009).
  - [7] S.A. Marvel and S. Strogatz, Chaos **19**, 013132 (2009).
  - [8] E.A. Martens, et. al., Phys.Rev.E **79**, 026204 (2009).
  - [9] A. Pikovsky and M. Rosenblum, Phys.Rev.Lett. **101**, 264103 (2009).
  - [10] L.M. Childs and S.H. Strogatz, Chaos **18**, 043128 (2009).
  - [11] P. So, B.C. Cotton and E. Barreto, Chaos **18**, 037114 (2009).
  - [12] D. Pazo and E. Montbrio, Phys.Rev.E **80**, 046215 (2009).
  - [13] A. Ghosh, D. Roy and V.K. Jirsa, Phys.Rev.E **80**, 041930 (2009).
  - [14] S. Watanabe and S.H. Strogatz, Phys.Rev.Lett. **70**, 2391 (1993); Physica D **74**, 197 (1994).
  - [15] S.A. Marvel, R.E. Mirollo and S.H. Strogatz, Chaos **19**, 043104 (2009).